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Heat transfer in turbulent fluids—I. Pipe flow

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Abstract—The expression for turbulent Prandtl number obtained from the renormalization group procedure is used to describe the process of heat transfer in turbulent pipe flow. The results are in a good agreement with experimental data over the entire range of experimentally accessible Prandtl numbers, $10^{-2} < \sigma_0 < 10^6$.

1. INTRODUCTION

THE PROBLEM of heat conduction in turbulent flows has been under intensive study for more than half a century. Experimental data on velocity and temperature distributions have suggested many semi-empirical theories to describe the basic properties of the phenomenon.

It has long been realised that, if the Reynolds number is large enough and the Prandtl number $\sigma_0 = \nu_0/\kappa_0$ is not too small, the molecular diffusivity κ_0 does not play any role in the process of heat conduction or diffusion in turbulence. In this case, the temperature and velocity distributions have similar behavior in the wall region, both obeying the logarithmic law with the temperature profile

$$\langle T \rangle = \sigma_{\text{turb}} \frac{q}{b\rho c_p u_*} (\ln y + C).$$

Here $\langle\ \rangle$ denotes a horizontal average, y is the distance to the wall, q denotes the constant heat flux and c_p and u_* are the heat capacity and friction velocity, respectively. The Von Karman constant $b \approx 0.4$ and $\sigma_{\text{turb}} = \nu_{\text{turb}}/\kappa_{\text{turb}}$ is the ratio of turbulent viscosity to turbulent heat conductivity. According to the well-known Prandtl–Reynolds–Colburn analogy, the turbulent Prandtl number is nearly a universal constant: $\sigma_{\text{turb}} = 0.7$–$0.9$.

In the limiting case of small Prandtl number, the molecular diffusivity κ_0 cannot be neglected and the simple analogy between temperature and velocity distributions does not work. It is clear, however, that as $\sigma_0 \to 0$, the Nusselt number Nu [defined below as the dimensionless (based on the bulk temperature, see equation (28)) heat flux] satisfies $Nu \approx$ const. It is known from experiments that $Nu \approx 6.8$–7.0 in flows with constant heat flux through the wall while Nu is somewhat smaller in flows with constant wall temperature. To the best of our knowledge, there is no satisfactory theory describing heat conductivity in turbulent flow with low Prandtl number.

Many attempts have been made to find empirical and semi-empirical relations to describe turbulent heat transfer across a wide range of Prandtl and Reynolds numbers. More than 30 formulae of this kind have been reviewed by Reynolds [1] in 1975. In 1979, Gori *et al.* [2] concluded that there is no general way to describe turbulent heat transfer in low-Prandtl-number fluids for a wide range of Re. They suggested the following formula for the turbulent Prandtl number σ_{turb} when $Re < 1.7 \times 10^5$:

$$\sigma_{\text{turb}}^{-1} = 0.014 Re^{0.45} \sigma_0^{0.2} \times \{1 - \exp[-(0.014 Re^{0.45} \sigma_0^{0.2})^{-1}]\} \quad (1)$$

as proposed by Aoki [3] or

$$\sigma_{\text{turb}}^{-1} = (1 + 100 Pe^{-0.5})[(1 + 120 Re^{-0.5})^{-1} - 0.15] \quad (2)$$

as proposed by Reynolds [1]. The formula

$$\sigma_{\text{turb}} = 0.85 + 0.005 \sigma_0^{-1} \quad (3)$$

proposed by Jischa and Rieke [4], was suggested to represent the Reynolds number range $1.7 \times 10^5 < Re < 2.6 \times 10^5$; the constant $\sigma_{\text{turb}} = 0.85$ was used for $Re > 2.6 \times 10^5$. When relations (1)–(3) are used to predict the mean temperature field, they give reasonably accurate predictions of the Nusselt number (which is related to the wall gradient of the temperature profile). However, the full temperature profiles predicted on the basis of expressions (1)–(3) were less satisfactory.

In this work we apply a formula for the turbulent Prandtl number derived by Yakhot and Orszag [5] to describe heat transfer in pipe flows. It will be shown in Section 3 that the proposed relation between turbulent viscosity and turbulent heat conductivity gives accurate predictions of both Nusselt number and temperature distributions across an extremely wide range of Prandtl and Nusselt numbers.

2. FORMULAE FOR TURBULENT PRANDTL NUMBER

Here we present some of the basic ideas leading to an expression for the turbulent Prandtl number. The main steps of the renormalization group procedure are

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NOMENCLATURE

A	Van Driest parameter
B	proportionality coefficient in x-coordinate dependence of temperature, equation (20)
A_d	geometric factor, equation (11)
C	constant in the temperature profile, equation (30)
C_h	heat transfer coefficient
D	pipe diameter
G^0	bare propagator for velocity
Nu	Nusselt number (based on the bulk temperature), $C_h \sigma_0 Re$
Pe	Péclet number, $\sigma_0 Re$
R	radius of pipe
Re	Reynolds number based on the pipe diameter, $u_{av} D/\nu_0$
R_*	Reynolds number based on the friction velocity, $u_* R/\nu_0$
T	temperature
T_c	temperature at the center
T_w	temperature at the wall
T_*	characteristic temperature, $q/c_p \rho u_*$
T_+	dimensionless temperature, T/T_*
a, b	parameters in equations (12)–(14)
c_p	heat capacity
d	fixed-point parameter [5], 7 in this work
g^0	bare propagator for temperature
q	heat flux
r_+	wall coordinate, $r u_*/\nu_0$
u	mean velocity in x-direction
u_*	friction velocity
u_+	dimensionless velocity, u/u_*
v_i	components of velocity.

Greek symbols

α	inverse total Prandtl number
α_0	inverse molecular Prandtl number
Θ	r-dependent component of temperature
Θ_+	dimensionless Θ, Θ/T_*
σ	total Prandtl number, ν/κ
σ_0	molecular Prandtl number, ν_0/κ_0
ε	expansion parameter in RNG procedure
$\bar{\varepsilon}$	turbulent dissipation rate
κ	total diffusivity, $\kappa_0 + \kappa_{turb}$
κ_0	molecular diffusivity
λ	friction coefficient, $8\tau_w/\rho u_{av}^2$
λ_0	expansion parameter
Λ_f	integral scale of turbulence
ρ	fluid density
ν	total viscosity, $\nu_0 + \nu_{turb}$
ν_0	molecular viscosity
ν_+	dimensionless total viscosity, ν/ν_0.

outlined in the Appendix. The details of the calculations are given elsewhere [5]. In this paper, we are interested in application of the final result to the problem of heat transfer in turbulent flow in a pipe. This will be done in the next section.

The most distinguishing characteristic of a turbulent flow is approximate universality of the properties of scales much smaller than any integral scale L in the flow. The high Reynolds number turbulent flow is characterized by three different ranges of spatial scales. (1) For wave-numbers $k > \pi/L$ the energy spectrum is strongly anisotropic and is not characterized in any universal way. The integral scale reflects both geometry of the flow and the physico-chemical processes taking place there. (2) At much smaller scales, with wave-numbers satisfying $\pi/L \ll k < k_d = Re^{3/4} L^{-1}$, the velocity fluctuation spectrum is approximately given by the Kolmogorov energy spectrum $E(k) = C_k \bar{\varepsilon}^{2/3} k^{-5/3}$, with the Kolmogorov constant $C_k = 1.3$–2.3. (3) In the dissipation range $(k > k_d)$ the energy spectrum decreases exponentially with k.

Universality of the small scales can be formulated in the language of theoretical hydrodynamics: the fluid described by the Navier–Stokes equation

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu_0 \frac{\partial^2 v_i}{\partial x_j \partial x_j} \qquad (4)$$

$$\frac{\partial v_i}{\partial x_i} = 0$$

subject to initial and boundary conditions, is characterized at the small scales by the 5/3-Kolmogorov spectrum. This property does not depend on boundary conditions which are usually characterized at large scales. Boundary conditions can be considered from the viewpoint of small scales as a source of energy injected into the large scales which subsequently cascade to the small scales. Using the analogy with equilibrium statistical mechanics, in which the results are independent of the details of the interaction of the system with a heat bath, we replace (4) by the more general equation (5) and add the heat transfer equation (6):

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = f_i - \frac{\partial p}{\partial x_i} + \nu_0 \frac{\partial^2 v_i}{\partial x_j \partial x_j} \qquad (5)$$

$$\frac{\partial v_i}{\partial x_i} = 0$$

$$\frac{\partial T}{\partial t}+v_i\frac{\partial T}{\partial x_i}=\kappa_0\frac{\partial^2 T}{\partial x_i\,\partial x_i} \tag{6}$$

where $\mathbf{f}$ is the random force (noise) chosen to generate the velocity field $\mathbf{v}$ described by the Kolmogorov spectrum in the limit of large wave-vectors (small scales).

It has been shown by Yakhot [8] that the Gaussian random force $\mathbf{f}$ characterized by the correlation function:

$$\langle f_i(k,\omega)f_j(k',\omega')\rangle \approx \bar{\varepsilon}k^{-3}P_{ij}(k)\delta(k+k')\delta(\omega+\omega') \tag{7}$$

with $P_{ij}(k)=\delta_{ij}-k_ik_j/k^2$, generates small-scale velocity fluctuations characterized by the Kolmogorov spectrum. The parameter $\bar{\varepsilon}$ in (7) denotes the dissipation rate of the turbulent energy per unit mass of the fluid and relates the force $\mathbf{f}$, acting on small scales, to the energy input taking place at large scales.

This fact is the basis for using the random force (7) for elimination of small scales in the construction of either turbulent sub-grid or transport models. The renormalization-group method (RNG) was developed for an infinite, homogeneous medium by Forster *et al.* [6], Martin and DeDominics [7] and Yakhot [8]. In these works, $\bar{\varepsilon}$ has been treated as a given parameter characterizing the rate of stirring. In finite systems

$$\bar{\varepsilon}=\frac{1}{T}\frac{1}{V}\int \mathrm{d}t\int \varepsilon(x,t)\,\mathrm{d}x \tag{8}$$

$$\varepsilon(x,t)=\frac{v_0}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)^2. \tag{9}$$

In such systems, $\bar{\varepsilon}$ is a quantity that should be determined dynamically from the equations of motion with boundary and initial conditions applied. The basic ideas of the renormalization group procedure are given in the Appendix.

It has been shown by Yakhot and Orszag [9] that the Navier–Stokes equations for the mean velocity field $\bar{v}_i$ in which the fluctuating contributions are removed is:

$$\frac{\partial \bar{v}_i}{\partial t}+\bar{v}_j\frac{\partial \bar{v}_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\nu\frac{\partial \bar{v}_i}{\partial x_j}. \tag{10}$$

Here the total viscosity ν takes into account both molecular and turbulent contributions and is given by the following relation [9]:

$$\nu=\nu_0\left[1+H\left(\frac{3}{8}\frac{A_{\mathrm{d}}\bar{\varepsilon}}{\Lambda_{\mathrm{f}}^4\nu_0^3}-100\right)\right]^{1/3} \tag{11}$$

where the ramp function $H(x)=x$ if $x>0$ and $H(x)=0$ if $x<0$ and Λ_{f} is the inverse integral scale of turbulence [9]. The parameter $A_{\mathrm{d}}=(d^2-d)/[2d(d+2)]=0.333$ since $d=7$ for this problem. It has been shown by Yakhot and Orszag [5] that elimination of small scales from the equations (4)–(6) of a passive scalar leads to the following relation between the inverse total Prandtl number $\alpha=\sigma^{-1}$ and the total viscosity ν:

$$\left(\frac{\alpha-a}{\alpha_0-a}\right)^{\gamma}\left(\frac{\alpha+b}{\alpha_0+b}\right)^{1-\gamma}=\frac{\nu_0}{\nu} \tag{12}$$

where

$$\gamma=\frac{a+1}{a+b}$$

$$a=\left[-1+\left(1+8\frac{d+2}{d}\right)^{1/2}\right]\Big/2 \tag{13}$$

$$b=a+1.$$

For $d=7$, relation (12) becomes

$$\left(\frac{\alpha-1.1793}{\alpha_0-1.1793}\right)^{0.65}\left(\frac{\alpha+2.1793}{\alpha_0+2.1793}\right)^{0.35}=\frac{\nu_0}{\nu}. \tag{14}$$

The result (14) expresses the inverse total Prandtl number α as a function of total viscosity ν and is the main result to be studied in this paper.

According to (11), the turbulent viscosity is itself a function of the distance from the wall since Λ_{f} must be associated with the distance to the wall. One sees that in the region of fully developed turbulence where $\nu_0/\nu \ll 1$, the total Prandtl number $\sigma=\alpha^{-1}=0.8476$, which is in a good agreement with available experimental data $\sigma=0.7$–0.9 (see Landau and Lifshitz [10] and Monin and Yaglom [11]). Close to the wall where $\nu\approx\nu_0$, one finds from (14) that $\sigma\approx\nu_0/\kappa_0$. Thus, the equation of motion for the mean temperature can be written as:

$$\frac{\partial T}{\partial t}+\bar{v}_i\frac{\partial T}{\partial x_i}=\frac{\partial}{\partial x_i}\kappa\frac{\partial T}{\partial x_i} \tag{15}$$

where $\kappa=\alpha\nu$ is determined from (14). The dynamics of diffusion of a passive scalar is governed by the set of equations (10), (11), (14), (15).

We emphasize that these results do not include any experimentally adjustable parameters.

3. HEAT CONDUCTIVITY IN PIPE FLOW

Here we apply the results presented in the previous section to describe the process of heat transfer in turbulent flow through a pipe of radius R. The problem can be formulated in terms of the stationary Navier–Stokes equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\nu\frac{\partial u}{\partial r}\right)=\frac{\partial p}{\partial x} \tag{16}$$

and the heat transfer equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa\frac{\partial T}{\partial r}\right)=u\frac{\partial T}{\partial x} \tag{17}$$

where ν and κ are total viscosity and diffusivity, respectively. The parameters ν and κ include both molecular and turbulent contributions. The total Prandtl number $\sigma=\nu/\kappa$ is determined from relation (14).

We introduce the friction velocity $u_*=(\tau_{\mathrm{w}}/\rho)^{1/2}$, wall coordinate r_+, nondimensional velocity u_+, and nondimensional total viscosity ν_+:

$$r_+ = r\frac{u_*}{\nu_0}, \quad u_+ = u/u_*, \quad \nu_+ = \nu/\nu_0. \qquad (18)$$

The equation of motion now has the nondimensional form:

$$\frac{1}{r_+}\frac{\partial}{\partial r_+}\left(r_+\nu_+\frac{\partial u_+}{\partial r_+}\right) = -\frac{2}{R_*} \qquad (19)$$

where $R_* = u_*R/\nu_0$ is the Reynolds number based on the friction velocity.

We consider heat transfer in a pipe with constant heat flux through the wall. In this case it is convenient to introduce a new variable Θ defined as:

$$T(x,r) = \Theta(r) + Bx. \qquad (20)$$

Substituting (20) into (17) yields an equation for $\Theta(r)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa\frac{\partial\Theta}{\partial r}\right) = uB. \qquad (21)$$

The parameter B can be expressed in terms of the imposed constant heat flux. Integrating (21) over r and using the fact that

$$\frac{\partial\Theta}{\partial r} = 0 \quad \text{at } r = 0 \qquad (22)$$

we obtain:

$$B = -\frac{4q}{c_p\rho Re\nu_0} \qquad (23)$$

where

$$q = -c_p\rho\kappa_0(\partial\Theta/\partial r)_{r=R} \qquad (24)$$

and the Reynolds number

$$Re = \frac{2}{\nu_0 R}\int_0^R u(r)r\,dr. \qquad (25)$$

Using relations (23)–(25) the heat equation (21) can be written in the nondimensional form:

$$\frac{1}{r_+}\frac{\partial}{\partial r_+}\left(r_+\alpha\nu_+\frac{\partial\Theta_+}{\partial r_+}\right) = -\frac{2u_+}{Re} \qquad (26)$$

where $\Theta_+ = \Theta/T_*$ and α is given by relation (14). The parameter T_* is defined as follows:

$$T_* = \frac{q}{c_p\rho u_*}. \qquad (27)$$

Using the above notation, the Nusselt number is given by

$$Nu = C_h\sigma_0 Re \qquad (28)$$

where

$$C_h = R_*^2\bigg/\left(2\int_0^{R_*}\Theta_+u_+r_+\,dr_+\right).$$

To describe heat transfer in turbulence, one needs an expression for the coefficient of heat conductivity which takes into account both molecular and turbulent contributions to the heat transfer process. The theory leading to relation (14) determines the turbulent diffusivity in terms of the laminar transport coefficients and the turbulent viscosity. In particular, it describes the interaction between molecular and turbulent transport, an effect of much significance at low Re and σ_0. Thus, the determination of turbulent heat transfer from (14) requires reliable data on turbulent viscosity. Such data can be found either from theory or from analysis of experimental data on velocity profiles in pipe flow.

In the present work we are interested exclusively in demonstrating the power of the 'universal' relation (14) provided the expression for turbulent viscosity is known. Thus, we adopt the *ad hoc* model [12] for the dimensionless total viscosity ν_+:

$$\nu_+ = 1 + 0.41y_+[1-\exp(-y_+^2/A^2)], \quad A = 26$$

when the distance to the wall $y_+ < 50$. The turbulent viscosity for $y_+ > 50$ is that derived from the differential k–$\bar{\varepsilon}$ model of Yakhot and Orszag [5]. The model viscosity and mean velocity profiles obtained by integrating the equation of motion (19) using this viscosity are presented in Figs. 1 and 2. The friction coefficient λ defined by $\tau_w = \lambda\rho u_{av}^2/8$ so $\lambda = 32(R_*/Re)^2$ is plotted in Fig. 3. It is apparent that the agreement with experimental data is very good.

The equation of motion (19) and heat equation (26)

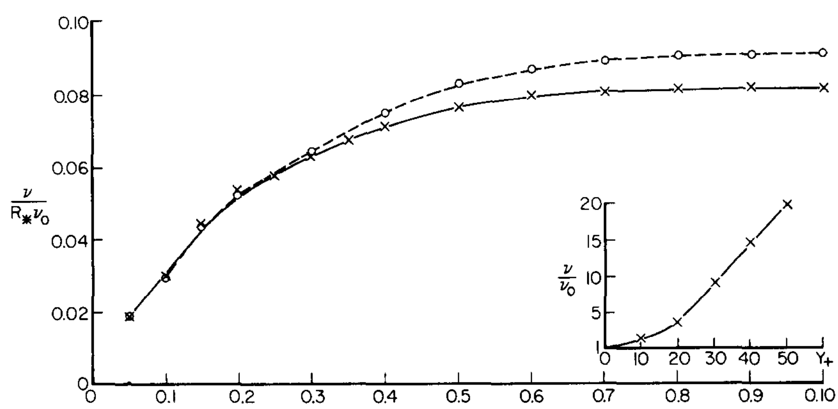

FIG. 1. Viscosity distribution in a pipe adopted in this work: × $Re = 40{,}000$; ○ $Re = 346{,}000$.

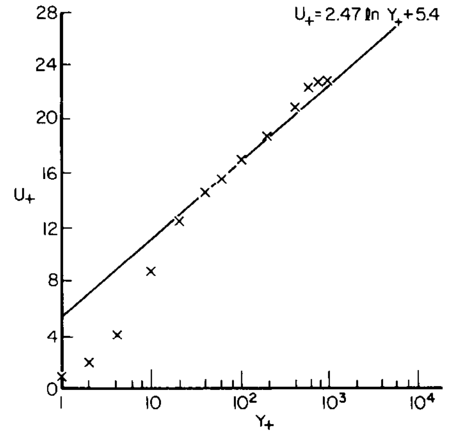

FIG. 2. Calculated dimensionless velocity profile, $u_+ = u/u_*$; $Re = 40{,}000$.

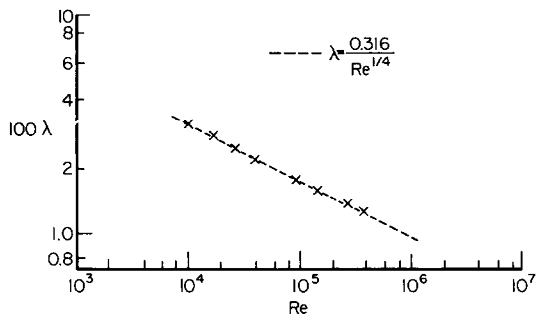

FIG. 3. Friction coefficient λ for turbulent pipe flow: × results of calculation based on the model viscosity from Fig. 1; ----- Blasius formula.

have been integrated using the model viscosity from Fig. 1 and α from the relation (14). The results are presented in Figs. 4–10 for various Prandtl (σ_0) and Reynolds (Re) numbers.

In Fig. 4, we plot the calculated and measured temperature profiles for air flow in a pipe. As we can see from Fig. 4, the agreement between the experimental data and the results of calculations for $\sigma_0 = 0.7$ is very good. The calculated Nusselt number for air flow ($\sigma_0 = 0.7$) is compared in Fig. 5 with the

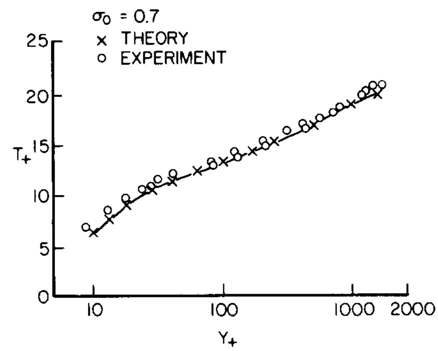

FIG. 4. Dimensionless temperature profile $T_+ = T/T_*$: × results of calculations; ○ experimental data [11].

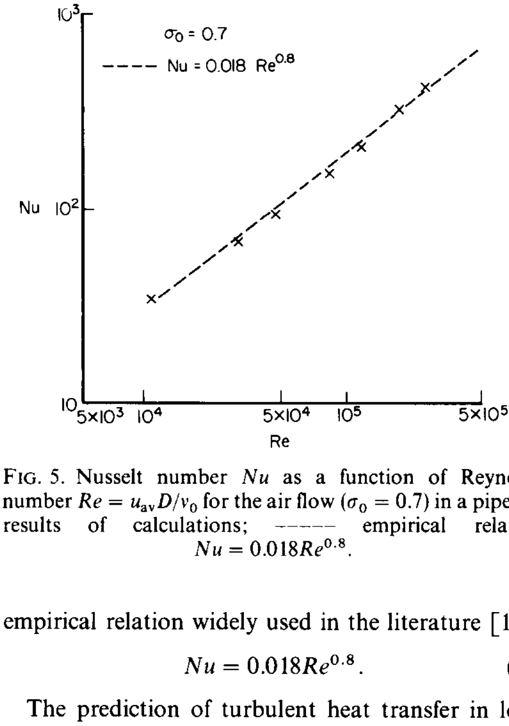

FIG. 5. Nusselt number Nu as a function of Reynolds number $Re = u_{av}D/\nu_0$ for the air flow ($\sigma_0 = 0.7$) in a pipe: × results of calculations; ----- empirical relation $Nu = 0.018Re^{0.8}$.

empirical relation widely used in the literature [11]:

$$Nu = 0.018Re^{0.8}. \qquad (29)$$

The prediction of turbulent heat transfer in low-Prandtl-number flow is a most difficult test for the model. In Fig. 6, the calculated temperature profiles in liquid mercury ($\sigma_0 = 0.02$) and in the NaK eutectic ($\sigma_0 = 0.029$) are plotted for pipe flow at $Re = 149{,}000$.

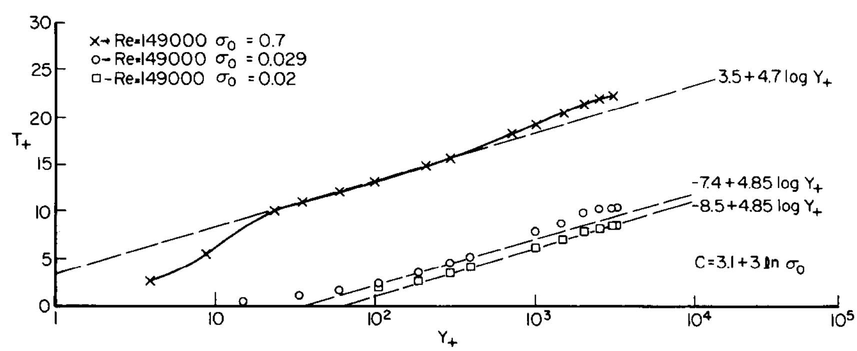

FIG. 6. Dimensionless temperature (T_+) profiles in turbulent flow in a pipe at $Re = 149{,}000$: × air ($\sigma_0 = 0.7$); ○ NaK eutectic ($\sigma_0 = 0.029$); □ mercury ($\sigma_0 = 0.02$); —— from ref. [13].

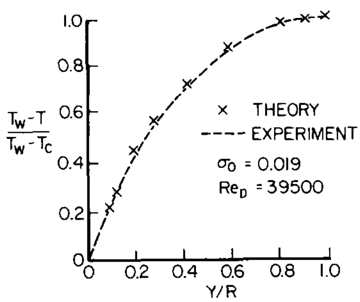

FIG. 7. Temperature defect $(T_w - T)/(T_w - T_c)$ distribution in turbulent flow in a pipe: × results of calculation for NaK eutectic ($\sigma_0 = 0.019$); ----- experimental data of ref. [13].

It is apparent that some fraction of the temperature profile ($y_+ = 100$) can be approximated by the logarithmic law:

$$T_+ = C + 4.85 \log y_+ \tag{30}$$

where the constant C can be found from the relation:

$$C = 3.1 + 3 \ln \sigma_0. \tag{31}$$

A relation very similar to (31) has been obtained from analysis of experimental data [11].

It should be mentioned that when the Prandtl number is small ($\sigma_0 = 0.02$) the logarithmic part of the temperature profile appears only at high Reynolds numbers: $Re > 10^5$. For $Re < 10^5$, no part of the temperature profile can be approximated by the relation (30).

The results of our calculations may be compared with the experimental data of Buhr *et al.* [13]. The temperature profile measured in the NaK eutectic flow ($\sigma_0 = 0.019$) at $Re \approx 4 \times 10^4$ is compared with the present results in Fig. 7. In Fig. 8, we plot calculated and experimental data for the temperature profiles in several low-Prandtl-number fluids at different Reynolds numbers in the range $3 \times 10^4 < Re < 3.5 \times 10^5$. Again, the agreement between the results of calculations and experimental data is very good. The Nusselt number Nu is plotted as a function of the Péclet number $Pe = \sigma_0 Re$ in Fig. 9. At low Pe, the

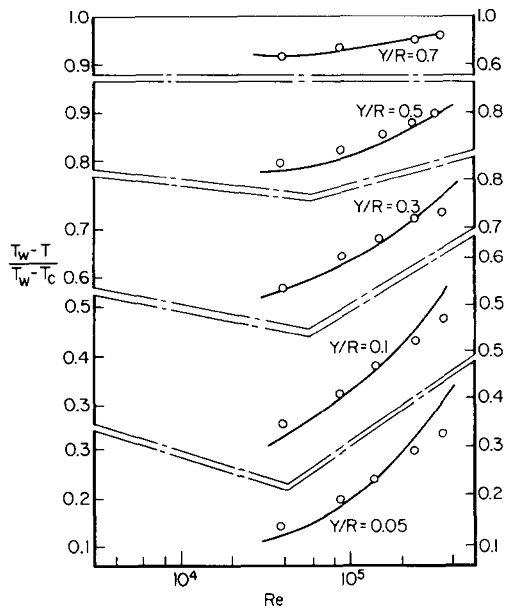

FIG. 8. Temperature defect $(T_w - T)/(T_w - T_c)$ distribution in turbulent flow in a pipe as a function of Reynolds number Re: ○ result of calculation; —— experimental data of ref. [13].

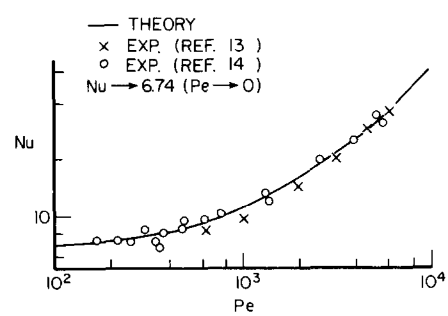

FIG. 9. Nusselt number Nu as a function of Péclet number Pe. $Nu \approx 6.74$ when $Pe \approx 0$.

results of numerical calculations give $Nu \approx 6.74$. This is very close to the experimentally observed [13, 14] limiting Nusselt number $Nu \approx 6.8$–7.0.

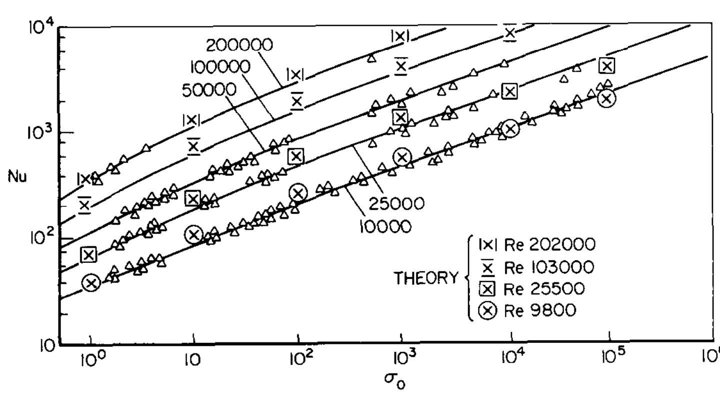

FIG. 10. Nusselt number Nu as a function of Prandtl number σ_0. Experimental data are taken from ref. [11].

Another test of both the relation (14) and of the model for turbulent viscosity adopted in this work is the prediction of heat transfer in high-Prandtl-number fluids. In this case, the molecular heat diffusivity is very low and the heat transfer process is determined entirely by the turbulent eddy diffusivity. The results of calculations are compared with experimental data in Fig. 10. The agreement with the results of measurements [11] is very good across a wide range of Prandtl and Reynolds numbers, $1 < \sigma_0 < 10^6$ and $2.5 \times 10^4 < Re < 2 \times 10^5$.

We conclude that the relation (14) can be used for the accurate description of turbulent heat transfer throughout the entire range of experimentally accessible Prandtl numbers, which vary over eight orders of magnitude, i.e. $10^{-2} < \sigma_0 < 10^6$.

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Note added in proof—In more recent work on the development of the RNG method (see ref. [5]), we have found that the proper technique is to evaluate all constants at the physical dimension $d = 3$ to lowest order in an expansion in powers of ε rather than at the critical dimension $d = 7$. This modification changes the turbulent Prandtl number to 0.7179 and changes the results presented here by several percent.

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APPENDIX

In this Appendix we will introduce the scale elimination procedure leading to renormalization of molecular viscosity ν_0 and molecular diffusivity κ_0. Using the incompressibility condition we write the equations of motion for the Fourier components of velocity $v_l(k, \omega)$ and passive scalar $T(k, \omega)$ as (see refs. [5–9]):

$$v_l(\hat{k}) = G^0(\hat{k}) f_l(\hat{k}) - \frac{i}{2} G^0(\hat{k}) P_{lmn}(k) \int v_m(\hat{q}) v_n(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \quad \text{(A1)}$$

$$T(\hat{k}) = -i g^0(\hat{k}) k_l \int v_l(\hat{q}) T(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}; \quad \hat{k} = (\mathbf{k}, \omega) \quad \text{(A2)}$$

where d is the dimensionality of the space

$$G^0(\hat{k}) = (-i\omega + \nu_0 k^2)^{-1} \quad \text{(A3)}$$

$$g^0(\hat{k}) = (-i\omega + \kappa_0 k^2)^{-1} \quad \text{(A4)}$$

and the random force $\mathbf{f}$ is given by the correlation function:

$$\langle f_i(\hat{k}) f_j(\hat{k}') \rangle = (2\pi)^{d+1} 2 D_0 k^{-y} P_{ij}(k) \delta(\hat{k}+\hat{k}'). \quad \text{(A5)}$$

The projection operator P_{lmn} is defined as $P_{lmn}(k) = k_m P_{ln}(k) + k_n P_{lm}(k)$. Here $y = 3$ and $D_0 \propto \bar{\varepsilon}$.

The equations of motion (A1) and (A2) are defined on the domain $0 < k \leqslant \Lambda$. The RNG procedure consists of two steps. First, we write equations in terms of the velocity field decomposition on two components $v^>(\hat{k})$ and $v^<(\hat{k})$ defined on the intervals $\Lambda e^{-l} \leqslant k \leqslant \Lambda$ and $0 < k < \Lambda e^{-l}$, respectively ($\lambda_0 = 1$):

$$v_l(\hat{k}) = G^0 f_l(\hat{k}) - \frac{i\lambda_0}{2} G^0 P_{lmn}(k) \int [v_m^<(\hat{q}) v_n^<(\hat{k}-\hat{q}) + 2 v_m^>(\hat{q}) v_n^<(\hat{k}-\hat{q}) + v_m^>(\hat{q}) v_n^>(\hat{k}-\hat{q})] \frac{d\hat{q}}{(2\pi)^{d+1}} \quad \text{(A6)}$$

$$T(\hat{k}) = -i\lambda_0 k_l g^0(\hat{k}) \int [v_l^<(\hat{q}) T^<(\hat{k}-\hat{q}) + v_l^>(\hat{q}) T^<(\hat{k}-\hat{q}) + v_l^<(\hat{q}) T^>(\hat{k}-\hat{q}) + v_l^>(\hat{q}) T^>(\hat{k}-\hat{q})] \frac{d\hat{q}}{(2\pi)^{d+1}}.$$

In order to eliminate modes from the interval $\Lambda e^{-l} < k < \Lambda$, all terms $v_l^>(\hat{k})$, $T^>(\hat{k})$ should be removed by repeated substitution of (A6) for $v^>$, $T^>$ back into (A6). This generates infinite expansions for $v^<$, $T^<$ in powers of λ_0 in which $v^>$, $T^>$ do not formally appear. Next, averages are taken over the part of the random force $f^>$ belonging to the strip $\Lambda e^{-l} < k < \Lambda$. This procedure formally eliminates the modes $\Lambda e^{-l} < k < \Lambda$ from the problem.

It follows from (A6) that, after removing the modes $\Lambda e^{-l} < k < \Lambda$, the equation of motion for $v^<$, $T^<$ can be written up to second order in λ_0 as:

$$\begin{aligned}(-i\omega + \nu_0 k^2) v_l^<(\hat{k}) &= f_l(\hat{k}) - \frac{i\lambda_0}{2} P_{lmn}(k) \int f_n^>(\hat{q}) f_m^>(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} - \frac{i\lambda_0}{2} P_{lmn}(k) \\ &\times \int v_m^<(\hat{q}) v_n^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} + 4\left(\frac{i\lambda_0}{2}\right)^2 2 D_0 P_{lmn}(k) \\ &\times \int |G^0(\hat{q})|^2 G^0(\hat{k}-\hat{q}) P_{n\mu q}(k-q) P_{m\mu}(q) q^{-y} v_q(\hat{k}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ &+ O(v^<)^3. \end{aligned} \quad \text{(A7)}$$

The equation for $T^<(\hat{k})$ is:

$$\begin{aligned}(-i\omega + \kappa_0 k^2) T^<(\hat{k}) &= -i\lambda_0 k_l \int v_l^<(\hat{q}) T^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} - 2\lambda_0^2 D_0 T^<(\hat{k}) k_l k_n \\ &\times \int |G^0(\hat{q})|^2 g^0(\hat{k}-\hat{q}) P_{ln}(q) \frac{d\hat{q}}{(2\pi)^{d+1}}. \end{aligned} \quad \text{(A8)}$$

When the $O(\lambda_0^2)$ terms on the RHS of (A7) and (A8), which are $O(k^2 v)$ and $O(k^2 T)$, respectively, are moved to the LHS, it gives corrections to the bare viscosity $\nu_0 k^2$ and diffusivity $\kappa_0 k^2$:

$$\Delta\nu = A_d \frac{\lambda_0^2 D_0}{\nu_0^2 \Lambda^4} \frac{e^{\varepsilon l} - 1}{\varepsilon}$$

and

$$\Delta\kappa = K_d \frac{\lambda_0^2 D_0}{\nu_0(\kappa_0+\nu_0)} \frac{e^{\varepsilon l}-1}{\varepsilon}$$

where

$$\varepsilon = 4+y-d$$

$$A_d = \frac{d^2-d-\varepsilon}{2d(d+2)} \frac{S_d}{(2\pi)^d}$$

$$K_d = \frac{d-1}{d} \frac{S_d}{(2\pi)^d}; \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

The parameter $d = 7$ at the fixed point. Thus, elimination of small scales leads to renormalization of viscosity and diffusivity. The second step of the procedure consists of iterating the scale-elimination procedure. This leads to the results given in Section 2.

TRANSFERT THERMIQUE DANS LES ECOULEMENTS TURBULENTS— I. ECOULEMENT DANS UN TUBE

Résumé—L'expression du nombre de Prandtl turbulent obtenue à partir d'une procédure de groupe de renormalisation est utilisée pour décrire le mécanisme du transfert thermique dans l'écoulement turbulent dans un tube. Les résultats sont en bon accord avec des données expérimentales dans le domaine des nombres de Prandtl $10^{-2} < \sigma_0 < 10^6$ accessibles expérimentalement.

WÄRMEÜBERGANG IN TURBULENTEN FLUIDEN—I. ROHRSTRÖMUNG

Zusammenfassung—Zur Beschreibung des Wärmeübergangs bei turbulenter Rohrströmung wird der Ausdruck für die turbulente Prandtl-Zahl verwendet, den man aus der Renormalisations-Gruppen-Prozedur erhält. Die Ergebnisse stimmen mit experimentellen Daten im gesamten Bereich der experimentell verfügbaren Prandtl-Zahlen, $10^{-2} < \sigma_0 < 10^6$, gut überein.

ТЕПЛОПЕРЕНОС В ТУРБУЛЕНТНЫХ ЖИДКОСТЯХ. ТЕЧЕНИЕ В ТРУБЕ

Аннотация—Для описания теплопереноса при турбулентном течении в трубе используется выражение для турбулентного числа Прандтля, полученное методом ренормализационной группы. Результаты находятся в хорошем соответствии с экспериментальными данными для всего диапазона значений числа Прандтля, $10^{-2} < \sigma_0 < 10^6$.